Asymptotic Blowup Solutions in MHD Shell Model of Turbulence

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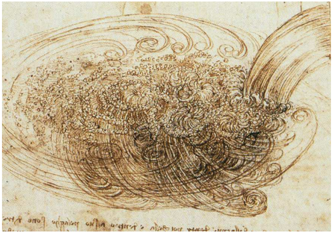
Turbulence

- Chaotic flow in a large range of scales;
- Natural phenomena for turbulent MHD flow;
  - Stellar formation;
  - Dynamo effect;
  - Solar weather;



Motivating question

- Problems:
  - Flow equations usually form a system of nonlinear differential equations with very rich behaviour, acting over an immense number of scales;
  - Are there solutions for every set of initial/boundary conditions? Are these solutions well defined for all time, i.e., **is there singularity formation in finite time (blowup)?**
- The existence of **blowup is and open problem** even in simple flow, such as 2D convective flow and 3D ideal flow.

The incompressible MHD equations

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} - \nu \nabla^2 \mathbf{v} = -(\mathbf{v} \cdot \nabla)\mathbf{v} + (\mathbf{b} \cdot \nabla)\mathbf{b} - \nabla p, \\ &\frac{\partial \mathbf{b}}{\partial t} - \eta \nabla^2 \mathbf{b} = \nabla \times (\mathbf{v} \times \mathbf{b}), \\ &\nabla \cdot \mathbf{v} = 0 \quad , \quad \nabla \cdot \mathbf{b} = 0, \end{aligned} \tag{1}$$

v and **b** are the velocity and induced magnetic fields;
p is the (magnetic and kinetic) pressure;
The density ρ has been taken as one.
These equations follow from the Navier-Stokes equation taking into account the Lorentz force and from Maxwell equations.

- The nonlinear terms on the right-hand side redistribute magnetic and kinetic energy among the full range of scales of the system.
- Three-dimensional systems have three ideal quadratic invariants, the total energy (E), the total correlation (C) and total magnetic helicity (H) given as follows:

$$E = \frac{1}{2}\int(\mathbf{v}^2 + \mathbf{b}^2)d^3x,$$
$$C = \int \mathbf{v}\cdot\mathbf{b}d^3x, \quad (2)$$
$$H = \int \mathbf{a}\cdot(\nabla\times\mathbf{a})d^3x,$$

where $\mathbf{a} = \nabla\times\mathbf{b}$.

Shell Models

- Discretization of the Fourier space onto concentric spherical shell, $k_{n-1} \leq \|\mathbf{k}\| < k_n$, $k_n = k_0 h^n$;
- Scalar variables are assigned to each shell; these variables account for fluid velocity and magnetic field;

$$\begin{aligned} \frac{dv_n}{dt} &= k_n[\epsilon(v_{n-1}^2 - b_{n-1}^2) + v_{n-1}v_n - b_{n-1}b_n] \\ &\quad - k_{n+1}[v_{n+1}^2 - b_{n+1}^2 + \epsilon(v_n v_{n+1} - b_n b_{n+1})], \\ \frac{db_n}{dt} &= \epsilon k_{n+1}[v_{n+1}b_n - v_n b_{n+1}] + k_n[v_n b_{n-1} - v_{n-1}b_n]. \end{aligned} \tag{3}$$

ϵ is a model free parameter [Gloaguen, et.at.;1985].

- Conservation of Energy and Cross-correlation

$$E = \frac{1}{2}\sum(u_n^2 + b_n^2)\ , \qquad C = \sum u_n b_n.$$

Blowup

- Example 1:

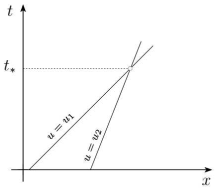
$$\frac{dy}{dt} = y^2. \tag{4}$$

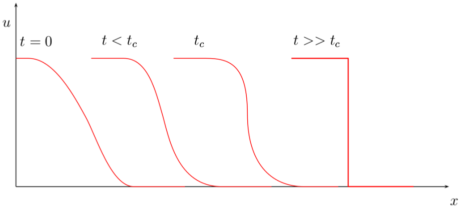
Solutions of this equation have the form

$$y(t) = (t_c - t)^{-1} \to \infty \quad \text{as} \quad t \to t_c. \tag{5}$$

- Example 2: inviscid Burgers equation

$$u_t + uu_x = 0 \tag{6}$$





Blowup criterion for MHD model

- We define the norms [Constantin,et al.; 2007]

$$\left\|v'\right\| = \left(\sum k_n^2 v_n^2\right)^{1/2}, \qquad \left\|v'\right\|_\infty = \sup_n k_n \left|v_n\right| . \tag{7}$$

- Blowup:

$$\left\|v'\right\| + \left\|b'\right\| \to \infty . \tag{8}$$

Theorem

Let $v_n(t)$ and $b_n(t)$ be a smooth solution of (3) regular for $0 \leq t < t_c$, where t_c is the maximal time of existence for such solution. Then, either $t_c = \infty$ or

$$\int_0^{t_c} \left\|v'\right\|_\infty dt = \infty. \tag{9}$$

Renormalization Scheme

Definition

Let τ be the renormalized time, implicitly defined [Dombre, Gilson; 1998] by

$$t = \int_0^{\tau} \exp\left(- \int_0^{\tau'} R(\tau'')d\tau'' \right) d\tau', \tag{10}$$

The renormalized velocity and magnetic variables are defined as

$$\begin{aligned} u_n &= \exp\left(- \int_0^{\tau} R(\tau')d\tau' \right) k_n v_n, \\ \beta_n &= \exp\left(- \int_0^{\tau} R(\tau')d\tau' \right) k_n b_n. \end{aligned} \tag{11}$$

Renormalized shell model

$$\frac{du_n}{d\tau} = -R(\tau)u_n + P_n \ , \qquad \frac{d\beta_n}{d\tau} = -R(\tau)\beta_n + Q_n \tag{12}$$

$$\begin{aligned} P_n &= \epsilon(h^2(u_{n-1}^2 - \beta_{n-1}^2) - u_n u_{n+1} + \beta_n \beta_{n+1}) \\ &\quad + h(u_{n-1}u_n - \beta_{n-1}\beta_n) - h^{-1}(u_{n+1}^2 - \beta_{n+1}^2), \\ Q_n &= \epsilon(u_{n+1}\beta_n - u_n\beta_{n+1}) + h(u_n\beta_{n-1} - u_{n-1}\beta_n). \end{aligned} \tag{13}$$

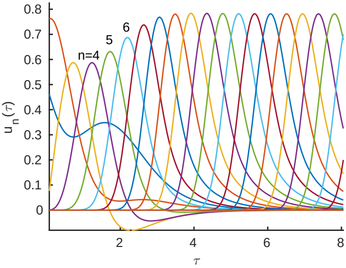
- $R(\tau)$ is found by imposing $\sum u_n^2 + \beta_n^2 = c$:

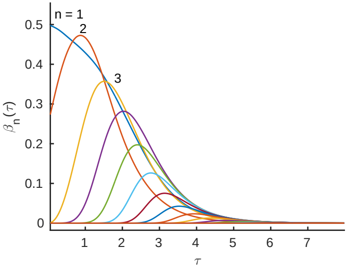
$$R(\tau) = \frac{\sum u_n P_n + \beta_n Q_n}{\sum u_n^2 + \beta_n^2} \tag{14}$$

Lemma: For any nontrivial initial conditions of finite ℓ^2-norm, a regular solution u_n and β_n of the renormalized system (12) exists and is unique for $0 \leq \tau < \infty$. $t(\tau) < t_c$, and blowup time is $t_c = \lim_{\tau \to \infty} t(\tau)$.

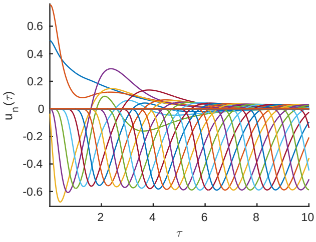
Types of solutions

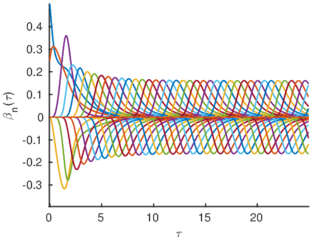
- Solutions of the renormalized model develop as different types of waves travelling towards larger shells
- Travelling wave solutions



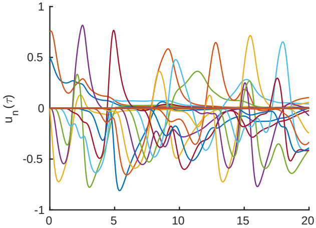


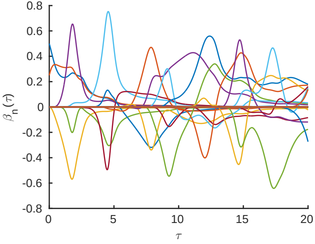
- Periodically pulsating wave solutions





- Chaotically pulsating wave solutions





Poincaré map

- We estimate the center n_w of a solution $w = (\ldots, u_n, u_{n+1}, \ldots, \beta_n, \beta_{n+1}, \ldots)$ as

$$n_w(\tau) = \sum n(u_n^2 + \beta_n^2) / \sum (u_n^2 + \beta_n^2) \tag{15}$$

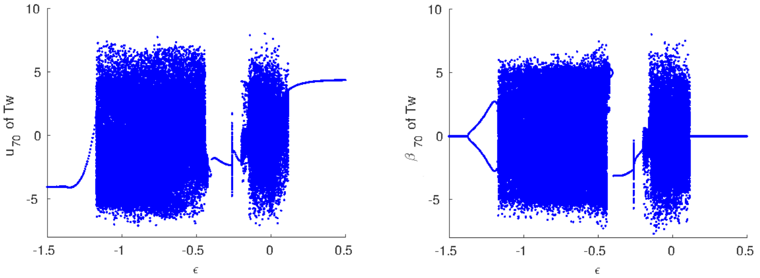
- We take the sequence τ_i as the times necessary for a solution center to travel by i shells

$$n_w(\tau_i) = n_w(0) + i \tag{16}$$

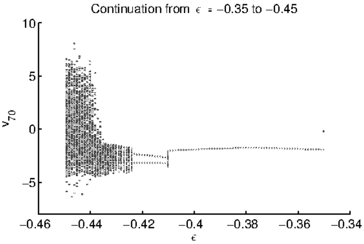
- We define a Poincaré map $\mathcal{P}$ as [Mailybaev; 2013]

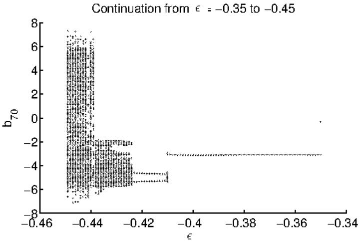
$$w' = \mathcal{P}w, \quad u_n' = u_{n+1}(\tau_1), \quad \beta_n' = \beta_{n+1}(\tau_1). \tag{17}$$

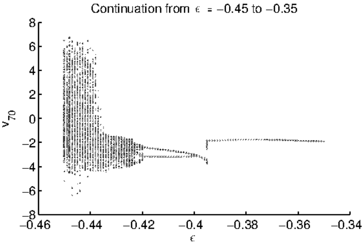
Bifurcation Diagram

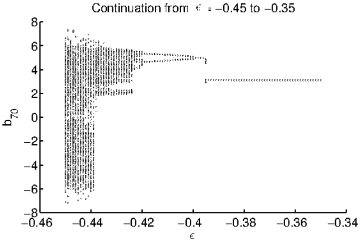


Multistability









Asymptotic travelling wave solution

- Let us first consider the case $b_n = 0$. For τ sufficiently big,

$$u_n(\tau) = aU(n - a\tau) \tag{18}$$

- We define

$$y = \frac{1}{\log h}\int_0^{1/a} R(\tau)d\tau, \quad V(t - t_c) = \exp\left(\int_0^{\tau} R(\tau)d\tau\right) U(-\tau) \tag{19}$$

where τ is related to t by (10) and R is given by (14).

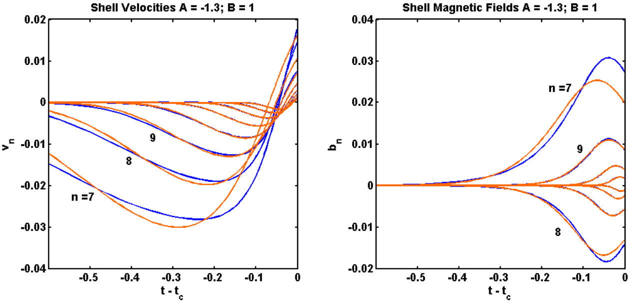
- **Theorem:** If $y > 0$, then solution $v_n(t)$ related to (18), for arbitrary posivite constant a, is given by

$$v_n(t) = ak_n^{y-1} V(ak_n^y(t - t_c)) \tag{20}$$

where the blowup time $t_c < \infty$ is given by

$$t_c = \int_0^{\infty} \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau' \tag{21}$$

Asymptotic Blowup Solution of Period (1,2)

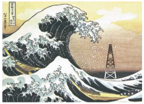


Conclusions

- We prove an analytic criterion for blowup;
- Our method constructs asymptotic blowup solutions;
  - These solutions are universal: depend only on the attractor, selected by the value of an invariant;
  - We show that there is blowup in the case of existence of these attractors;
- Asymptotic solutions give scaling laws near blowup, useful for other applications;
- Observation of competing blowup scenarios.
- Implications for MHD flow:
  - role of magnetic field in blowup;
  - dynamo effect in blowup.

Institutional support

- Masters in computational math and modelling (IMPA);
  - Scholarship from PRH-32 ANP/MCTI;



IMPA PRH/ANP-32

prh Programa de Recursos Humanos da ANP

anp Agência Nacional do Petróleo, Gás Natural e Biocombustíveis